Diffusion and trapping on a one-dimensional lattice

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The properties of a particle diffusing on a one-dimensional lattice where at each site a random barrier and a random trap act simultaneously on the particle are investigated by numerical and analytical techniques. The combined effect of disorder and traps yields a decreasing survival probability with broad distribution (log normal). Exact enumerations, effective-medium approximation, and spectral analysis are employed. This one-dimensional model shows rather rich behaviors which were previously believed to exist only in higher dimensionality. The possibility of a trapping-dominated super-universal-class is suggested. [S1063-651X(96)03606-9]

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I. INTRODUCTION

One-dimensional models are widely used in the physics of disordered systems [1]. This is because on one hand they are often good representatives of higher-dimensional models and on the other hand they are much easier to handle. In recent decades the importance of investigating simple diffusion in the presence of disorder and trapping has been widely appreciated as a toy model for complex systems, including migrations of optical excitations [2], polymer physics [3], and diffusion-limited binary reactions [4]. See, e.g., [5] and [6] for an exhaustive review.

From the mathematical viewpoint the analytical solution of the diffusion problem of particles diffusing before getting completely absorbed by permanent traps has proved an extremely difficult task and only an asymptotic solution for the survival probability (to be defined below) in the presence of uncorrelated disorder could be given using sophisticated techniques [7]. A variation of this problem, in the presence of strongly correlated (percolating) disorder, was also recently numerically investigated [8,9], and it was observed that whenever the total survival probability is an erratically decreasing function (i.e., detailed balance is violated) new and unexpected behaviors such as enhanced diffusion, breaking of self-averaging, and emergence of Lifshitz tails [8,9] appear. All these latter phenomena were somehow believed to stem from the percolating lattice used to model the disorder.

Here we shall investigate a different problem, where walks are partially and randomly absorbed at *each* site of a one-dimensional lattice, thus leading to a nonconservation of the probability. From the physical point of view this model can mimic the partial absorption of a set of excitons wandering in mixed crystals. We shall show that despite the low dimensionality of the model, a rich variety of features qualitatively similar to the ones observed in the trapping in percolating disorder [8,9] can be found. The low dimensionality of the model, however, leaves open the possibility of a full

analytical solution which would extend the results presented here.

Diffusion on a one-dimensional hierarchical lattice [10] and multifractal characterization of the escape probability [11] were previously considered in this context.

Our aim here is then twofold. On one hand, we shall show that most of the pathological features of the model considered in Refs. [8,9] are also present in this simplified low-dimensional version. On the other hand, the results presented here can be regarded as a complement of earlier investigations [6,12] where the total probability is conserved, thus providing a direct test of the effect of the nonconservation of the probability.

The outline of the paper is as follows. In Sec. II we introduce the model and recall some well known general manipulations of disordered one-dimensional lattices. Section III contains a numerical solution of the master equation. Section IV contains an effective-medium approximation to the diffusion, which is shown to be inadequate. However, in Sec. V a heuristic argument patterned after Grassberger and Procaccia's similar reasoning for the Donsker-Varadhan case [7] provides an intuitive explanation of the numerical results. Section VI contains a detailed numerical investigation of the influence of trapping on the spectrum and the localization properties. Finally, in Sec. VII some conclusions are drawn up.

II. MODEL

Consider a particle moving on a one-dimensional lattice with random barriers and random trapping probability on each site. The master equation reads

$$P_{x_0,x}(t+1) = \gamma(1-\epsilon_x)P_{x_0,x}(t) + w_{x,x-1}P_{x_0,x-1}(t)$$
$$+ w_{x,x+1}P_{x_0,x+1}(t) \tag{1}$$

for the probability $P_{x_0,x}(t)$ of being at site x at time t, having started from site x_0 at the initial time $t=0$. In this nota-

tion $w_{x,y}$ is the hopping probability from site y to site x, $\epsilon_x = w_{x-1,x} + w_{x+1,x} < 1$, and $\gamma \in [0,1]$ is a parameter defining the sojourn probability, which can be continuously tuned from from 1 (no trapping) to 0 (full trapping). Thus at each time step a particle at site x can move to sites $x \pm 1$ with random probability $w_{x,x\pm 1}$, stay at x with sojourn probability $\gamma(1-\epsilon_x)$, and disappear with probability $(1-\gamma)(1-\epsilon_x)$. In the following we shall consider only the case of symmetric hopping probability ($w_{x,y} = w_{y,x}$) and an infinite lattice.

As is well known [1], when $\gamma = 1$ this model can be mapped into a variety of other one-dimensional models.

The (discrete) Laplace transform $\widetilde{P}_{x_0,x}(\omega)$ of the probability $P_{x_0,x}(t)$ satisfies an equation of motion which can be cast in the following Green equation for a tight binding (TB) Hamiltonian:

$$\sum_y [E - \mathcal{H}(\gamma)]_{x,y} G_{x_0,y}(E) = \delta_{x,x_0}, \qquad (2)$$

corresponding to the generalized TB Hamiltonian

$$\mathcal{H}_{x,y}(\gamma) = \Omega_x(\gamma)\delta_{y,x} - w_{x,y}(\delta_{y,x-1} + \delta_{y,x+1}), \qquad (3)$$

where we have defined $\Omega_x(\gamma) \equiv \gamma\epsilon_x + 1 - \gamma$ and $G_{x_0,x}(E) \equiv -\widetilde{P}_{x_0,x}(\omega)|_{\omega=-E}$. Here the (discrete) Laplace transform is defined as

$$\widetilde{P}_{x_0,x}(\omega) = \sum_{t=0}^{+\infty} \frac{P_{x_0,x}(t)}{(1+\omega)^{t+1}}. \qquad (4)$$

On the other hand, Eq. (1) can be written in the form

$$P_{x_0,x}(t+1) = \sum_y T_{x,y}(\gamma) P_{x_0,y}(t), \qquad (5)$$

where we have defined the transition matrix

$$T_{x,y}(\gamma) = \begin{cases} \gamma(1-\epsilon_x) & \text{if } x=y \\ w_{x,y} & \text{if } |x-y|=1 \\ 0 & \text{otherwise.} \end{cases} \qquad (6)$$

It is then easy to check that if $\gamma < 1$ (and thus $\sum_x T_{x,y} < 1$ for any x), then all the eigenvalues $\{\lambda_\alpha\}$ of T are *strictly* less than unity, i.e., $|\lambda_\alpha| < 1$ for any α, unlike the conserved case ($\gamma = 1$) where the fact that the maximum eigenvalue λ_M is nondegenerate and equal to 1 irrespective of the (finite) size of the system is ensured by the Perron-Frobenius theorem [13]. Equivalently, this means that the eigenvalues $\{E_\alpha\}$ of the Hamiltonian (3) are all *strictly* positive, i.e., $E_\alpha > 0$ for any α.

All the results presented in this paper were obtained using the discrete time equation (1). The continuum time counterpart of Eq. (1) would be, in the coutinuous time random walk (CTRW) formulation,

$$\partial_t P_{x_0,x}(t) = \hat{w}_{x,x-1}P_{x_0,x-1}(t) + \hat{w}_{x,x+1}P_{x_0,x+1}(t)$$

$$- \hat{\epsilon}_x P_{x_0,x}(t) - \hat{\beta}_x P_{x_0,x}(t), \qquad (7)$$

where we have indicated with $\hat{\beta}_x$ the absorption rate at site x and used the tildes to indicate the quantities which are rates. By balancing the gain, loss, and sojourn terms to unity, it is easy to see that $\hat{\beta}_x = (1-\gamma)(1-\hat{\epsilon}_x)$. However, it should be stressed that Eq. (7) is *not* the continuum limit of Eq. (1).

III. EXACT ENUMERATION

We have considered a distribution of the disorder given by

$$\rho(w) = 2^{1-\alpha}(1-\alpha)w^{-\alpha}\theta(w)\theta(1/2-w), \qquad (8)$$

where $\alpha \in (-\infty, 1)$. By varying the parameter α we can pass from *weak* disorder ($\alpha < 0$) to *strong* disorder ($\alpha \to 1$). The case $\alpha = 0$ corresponds to a uniform distribution and it is *marginal* in the sense that the inverse first moment is logarithmically divergent. We shall indicate with an overbar the average over the disorder (8) which is assumed to be *quenched*. The simplest quantity that one would like to compute is the mean-square displacement

$$\langle x^2(t) \rangle = \frac{\overline{\Sigma_x(x-x_0)^2 P_{x_0,x}(t)}}{\Sigma_x P_{x_0,x}(t)}, \qquad (9)$$

where it should be noted that the denominator must be included since the total probability is not conserved. The disorder average of the latter is called the *survival probability* $P_s(t)$ and it is a decreasing function of time, once we have chosen the initial conditions $P_s(0) = 1$. The return probability

$$P_0(t) = \frac{\overline{P_{x_0,x_0}(t)}}{\Sigma_x P_{x_0,x}(t)} \qquad (10)$$

is another interesting quantity to look at and it is again different from the *survival return* probability $P_s^0(t) = \overline{P_{x_0,x_0}(t)}$ in view of the nonconservation of the probability. A suitable normalization procedure [8] is to be used in order to avoid a quick underflow of the survival probability.

We numerically solved the master equation exactly up to $t = 1000$ for three representative values of the strength of the disorder, namely, $\alpha = -0.5$ (weak disorder), $\alpha = 0$ (marginal disorder), and $\alpha = 0.5$ (strong disorder). The lattice size was chosen sufficiently large (up to $N = 2^{18}$) to avoid finite size effects due to the boundaries. Our estimates are based on an average of three samples of 1000 different configurations each. Errors are statistical.

The case $\gamma = 1$ was previously studied in Ref. [6] and our numerical results are in perfect agreement with the one reported there. Indeed, we find

$$\langle x^2(t) \rangle = t^{2\nu}, \qquad (11)$$

where ν is the correlation length exponent, and

$$P_0(t) = t^{-d_s/2}, \qquad (12)$$

where d_s is the return probability exponent. Our estimates are $2\nu = 0.99 \pm 0.01$ and 0.68 ± 0.01 for $\alpha = 0$ and 0.5, respectively, which compare well with the expected values 1 and 2/3.
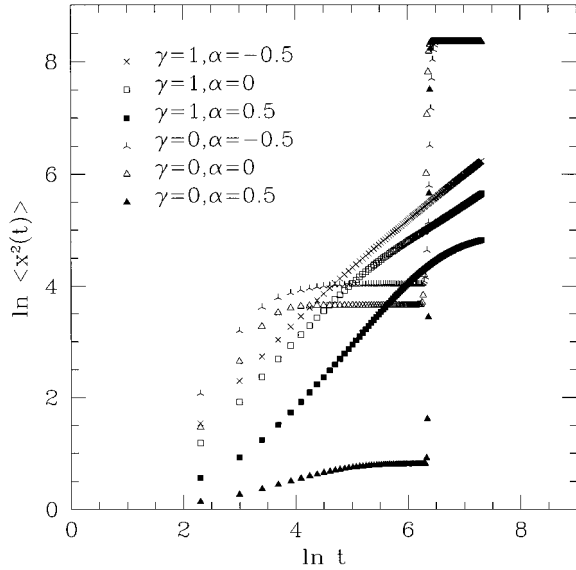
FIG. 1. Log-log plot of the mean square displacement $\langle x^2(t)\rangle$ for a single configuration in the cases $\gamma=1$ (no traps) and $\gamma=0$ (full trap) for representative values of the disorder ($\alpha=-0.5,0,0.5$). The $\gamma=0$ case appears to be independent of the strength of the disorder parameter α.

The case $\gamma=0$, corresponding to a full trapping, appears to be completely *trapping dominated*. Indeed, here the sample-to-sample fluctuations are enormous and they completely rule the diffusion. In order to have a feeling for this, we plotted in Fig. 1 the mean-square displacement for a single configuration obtained with the same initializing seed, in the cases $\gamma=1$ and 0 for various strengths of the disorder. The staircase behavior which can be observed in the case $\gamma=0$ is a consequence of the nonconservation of the probability. A similar effect occurs in the return probability. This rather peculiar feature was already observed in the model of Ref. [8]. Here, however, the extremely weak dependence of the behavior on the strength of the disorder is noteworthy.

Upon disorder averaging we find, in the $\gamma=0$ case, that the first moment [$\langle x(t)\rangle$] is zero as expected, and the second moment $\langle x^2(t)\rangle$ and the return probability $P_0(t)$ follow the behaviors (11) and (12), respectively.

Our best estimates for the exponents ν and d_s are reported in Table I. As mentioned above, the very similar result for the three cases of α is somewhat surprising, but it is in agreement with the aforementioned interpretation. Although the three values are not all within the numerical errors, we cannot rule out the possibility that the universality class is *independent* of the disorder choice. We also checked that all the intermediate cases $0<\gamma<1$ behave as in the $\gamma=0$ case after a transient time which depends on the value of γ. It should be

TABLE I. Summary of the exponents ν and d_s for the $\gamma=0$ case at three representative values of the strength of the disorder.

α	2ν	$d_s/2$
-0.5	1.24 ± 0.01	0.57 ± 0.01
0	1.25 ± 0.01	0.59 ± 0.01
0.5	1.26 ± 0.01	0.60 ± 0.01

stressed that d_s bears the meaning of the *spectral dimension* [6] only in the case $\gamma=1$.

Although we monitored the behavior of the survival probability and found that it decays as a stretched exponential (i.e., with an argument for the exponential which is a power of the time less than 1), we also found that the numerical value of the exponent is not very easy to pin down. This was to be expected: a similar feature occurs in the Donsker-Varadhan problem [14].

In view of these results, one expects both the survival probability and the survival return probability to be *non-self-averaging* quantities. We investigated the full probability distributions of both quantities. We find (see Fig. 2) that they rather accurately follow a log-normal distribution:

$$\mathcal{P}[X(t)] = \frac{1}{X\sqrt{2\pi\sigma_t^2}}\exp\left[\frac{-(\ln X - \lambda_t)^2}{2\sigma_t^2}\right], \quad (13)$$

where $X(t)=P_{x_0,x_0}(t)$ or $\Sigma_x P_{x_0,x}(t)$. Here λ_t and σ_t^2 are the mean and the variance of the distribution, respectively. If asymptotically ($t\gg1$) it happens that $\sigma_t^2\gg\lambda_t$ then self-averaging is *broken* (see the discussion in Ref. [8]). Both these quantities can be computed directly as first (λ_t) and second (σ_t^2) moments of the distribution but also indirectly by fitting the evolution of the log-normal function at various values of t. We find the following behavior for $t\gg1$:

$$\sigma_t^2 \sim t^{2\chi} \quad (14)$$

for the survival probability (and this defines the ''free-energy'' exponent χ), and a similar behavior for the survival return probability (which defines the analogous exponent χ_0). Our best estimates for these exponents are reported in Table II. On the other hand, we find that asymptotically $\lambda_t\sim t$ for both the survival and the survival return probability. Since in all cases $2\chi>1$ the self-averaging property is broken [8].

As a final remark we computed higher moments [$\langle x^4(t)\rangle$, $\langle x^6(t)\rangle$, etc.] in the attempt to find signatures of multifractality as suggested by the results of Ref. [11]. We found that all the moments were related to the second one irrespective of the strength of the disorder, thus ruling out the possibility of multifractal behavior in the size of the walk.

IV. EFFECTIVE-MEDIUM APPROXIMATION

In this section we will tackle the problem of solving Eq. (1) by using an effective-medium approximation (EMA). Although this approximation is known to fail in many situations [5], it nevertheless gives extremely accurate results for the problem described by (1) with $\gamma=1$ [6]. Here we shall carry the successful recipe given in [6] to the other extreme, namely, the case of full trapping ($\gamma=0$).

The equation of motion for the Laplace transform $\widetilde{P}_{x_0,x}(\omega)$ can be written in the form (taking $x_0=0$ for concreteness)

$$\mathcal{G}_{0,x}(\omega) = \mu_{x-1}(\omega)\mathcal{G}_{0,x-1}(\omega) + \mu_x(\omega)\mathcal{G}_{x+1}(\omega) + \delta_{x,0}, \quad (15)$$
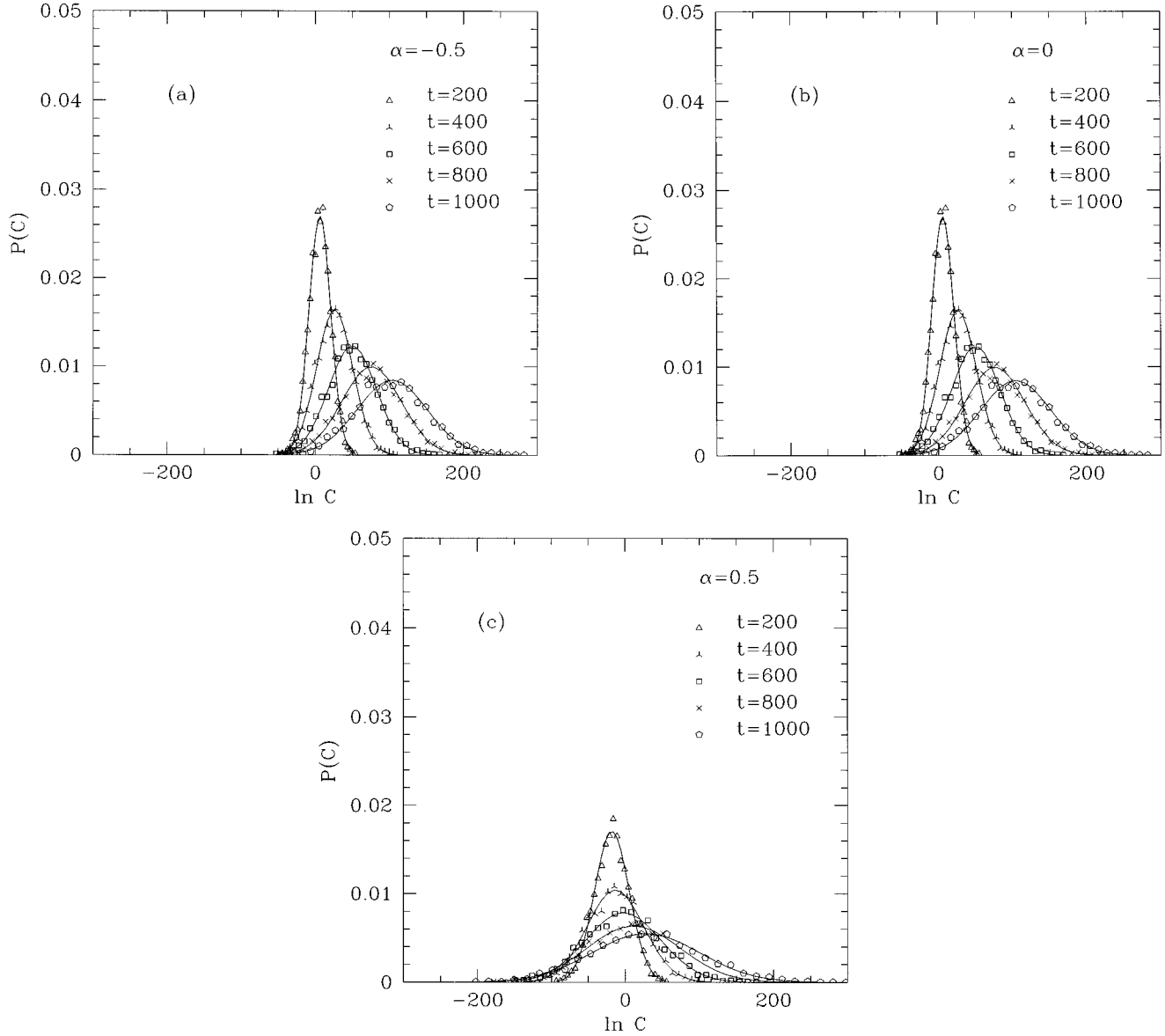
FIG. 2. Distribution density probability for the survival probability for time steps $t=200,400,600,800,1000$. The full lines correspond to the fitted log-normal distribution of the form (13). Shown are the cases $\alpha=-0.5$, 0, and 0.5 corresponding to (a), (b), and (c), respectively.

where $\mathscr{G}_{0,x}(\omega)=(1+\omega)\widetilde{P}_{0,x}(\omega)$ and where we have defined $\mu_x(\mu_{x-1})=w_{x,x+1}(w_{x-1,x})/(1+\omega)$. For $x>0$ and $x<0$ one can introduce the following new fields:

$$\phi_x^+(\omega)=\frac{\mu_{x-1}(\omega)\mathscr{G}_{0,x}(\omega)}{\mathscr{G}_{0,x-1}(\omega)}, \quad \phi_x^-(\omega)=\frac{\mu_x(\omega)\mathscr{G}_{0,x}(\omega)}{\mathscr{G}_{0,x-1}(\omega)},$$
(16)

TABLE II. Summary of the exponents χ and χ_0 defined in the text. The labels (D) and (I) mean direct evaluation and evaluation from the log-normal distribution, respectively. The values corresponding to a dash were considered unreliable due to the presence of a strong curvature in the preasymptotic regimes.

α	$\chi(D)$	$\chi_0(D)$	$\chi(I)$	$\chi_0(I)$
-0.5	0.70 ± 0.01	0.72 ± 0.01	0.72 ± 0.01	0.70 ± 0.01
0	0.73 ± 0.01	—	0.72 ± 0.01	0.70 ± 0.01
0.5	—	—	0.72 ± 0.01	0.70 ± 0.01

respectively. In this way the problem is reduced to first order and can be solved by continued fractions. For $x>0$ one finds the following recursions:

$$\phi_x^+(\omega)=\frac{\mu_{x-1}^2(\omega)}{1-\phi_{x+1}^+(\omega)}.$$
(17)

A similar equation can be found for $x<0$. It should be noted that the support for the $\phi^{+,-}$ has to be constrained in such a way that Eq. (17) be sensible. From Eq. (15) for the case $x=0$ and using Eq. (17) and the corresponding one for $x<0$ one easily finds that

$$\mathscr{G}_{0,0}(\omega)=\frac{1}{1-\phi_{+1}^+(\omega)-\phi_{-1}^-(\omega)}.$$
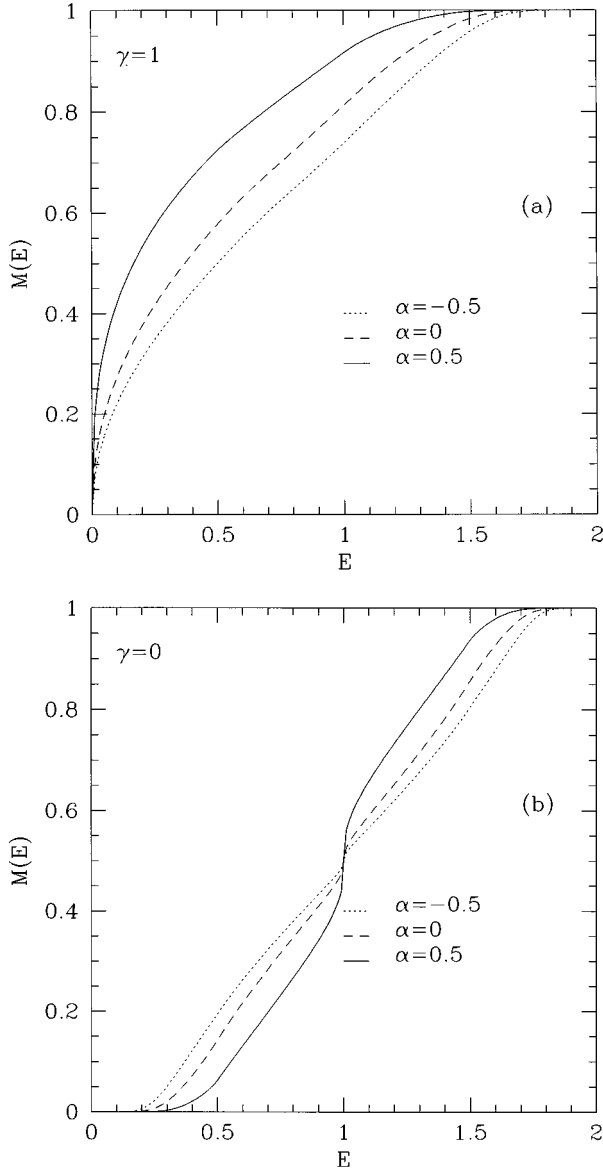(18)

The disorder average of Eq. (18) thus gives

FIG. 3. Integrated density of states on the whole spectrum $E \in [0,2]$ for $\gamma=1$ (a) and $\gamma=0$ (b) for the same strengths of disorder as in Fig. 2.

$$\overline{\mathscr{G}_{0,0}(\omega)} = \int d\phi_+ \int d\phi_- \Pi_\omega^+(\phi_+)\Pi_\omega^-(\phi_-)$$
$$\times \frac{1}{1-\phi_+-\phi_-}\,\theta(1-\phi_+-\phi_-), \qquad (19)$$

where the distributions $\Pi_\omega^+(\phi_+)$ and $\Pi_\omega^-(\phi_-)$ are given by

$$\Pi_\omega^+(\phi_+) = \int d\mu\, \rho(\mu) \int d\phi_+' \Pi_\omega^+(\phi_+')\delta\!\left(\phi_+ - \frac{\mu^2}{1-\phi_+'}\right),$$
$$(20)$$

and similarly for $\Pi_\omega^+(\phi_-)$. Now comes the EMA approximation:

$$\Pi_\omega^+(\phi_+) = \Pi_\omega^-(\phi_-) = \delta(\phi - \phi_e(\omega)), \qquad (21)$$

where $\phi_e(\omega)$ is an *effective* field to be found self-consistently. We can immediately show that the self-consistency equation decouples and its solution is given by

$$\phi_e(\omega) = \frac{1-\sqrt{1-4\mu_a^2(\omega)}}{2} \qquad (22)$$

$[\mu_a(\omega) = \mu_a/(1+\omega)]$ provided that the second moment of the distribution

$$\mu_a^2 \equiv \frac{\int d\mu\, \mu^2\rho(\mu)}{\int d\mu\, \rho(\mu)} \qquad (23)$$

exists. As a consequence,

$$\overline{\mathscr{G}_{0,0}(\omega)} = \frac{1}{\sqrt{1-4\mu_a^2(\omega)}}. \qquad (24)$$

Due to conditions imposed on the transition rates, we have that $\mu_a \leqslant 1/2$. If equality holds then the result is identical to the one obtained in the absence of disorder. If, however, $\mu_a < 1/2$ then by antitransforming back to the direct domain (in the continuum approximation), it is easy to show that the survival return probability behaves as

$$P_s^0(t) = e^{-t}I_0(2\mu_a t), \qquad (25)$$

where $I_0(z)$ is the zeroth-order Bessel function. Upon asymptotic expansion [15] the leading behavior for $t \gg 1$ is

$$P_s^0(t) = \frac{e^{-(1-2\mu_a)t}}{\sqrt{4\pi\mu_a t}}\left[1 + \frac{1}{16\mu_a t} + O\!\left(\frac{1}{t^2}\right)\right]. \qquad (26)$$

The result is *qualitatively* different from the numerical indication. This was to be expected in [5] that the EMA may not be capable of capturing the nuances of systems where large sample-to-sample fluctuations are present.

V. HEURISTIC ARGUMENT

In this section we shall present an intuitive argument yielding a stretched exponential for the survival probability in *qualitative* agreement with our numerical findings [16]. The argument is based on a similar one given by Grassberger and Procaccia [7] for the Donsker-Varadhan problem, and provides a plausible explanation of the irrelevance of the choice of the distribution for the hopping probability $\rho(w)$.

The main idea is that the maximum contribution to the survival probability comes from rare regions where the hopping probabilities are all very close to 1/2, say in the interval $[1/2-\epsilon,1/2]$ (with $0<\epsilon\ll1/2$) for the sake of the argument. If s is the number of sites of one of these regions, the typical number of steps τ necessary to explore the region is $\tau \sim s^{1/\nu}$. If there were no leaking of probability the decay of the survival probability in this region $P(t,s)$ would be exponential. However, during the time τ, there is a loss of probability of order ϵ^τ. The probability $\mathscr{P}(s)$ of finding such a region is $(2\epsilon)^s$. Therefore the survival probability is expected to have the form
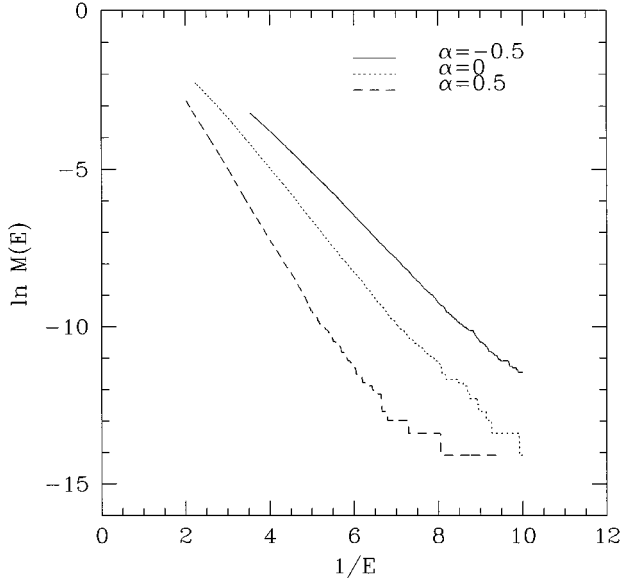
FIG. 4. Plot of the logarithm of the integrated density of states, $\ln M(E)$, versus $1/E$ for $E \to 0$ in the case $\gamma = 0$ (full trapping) for $\alpha = -0.5, 0, 0.5$. The behavior is qualitatively different from the case $\gamma = 1$ where $M(E)$ is a power law in E.

$$P_s(t) = \int_0^{+\infty} ds \; \mathscr{P}(s) P(t,s) \sim \int_0^{+\infty} ds \; \exp[-f_t(s)],$$
(27)

where for $\epsilon \ll 1$ we have that

$$f_t(s) = \frac{t}{s^{1/\nu}} + \lambda s^{1/\nu} + \lambda s,$$
(28)

where $\lambda = |\ln \epsilon|$. The integral can be carried out using the steepest descent method which yields the equation

$$t = \lambda s_0^{2/\nu} + \lambda \nu s_0^{(1+\nu)/\nu}$$
(29)

for the saddle point s_0. Since $\nu < 1$, at the leading order for $t \gg 1$, the solution is

$$s_0 \sim (1/\lambda)^{\nu/2} t^{\nu/2}.$$
(30)

Therefore, upon substitution in Eq. (27) we find at the leading order in $t \gg 1$

$$P_s(t) \sim \exp(-C\sqrt{t}),$$
(31)

where C is a constant, which is different from the Donsker-Varadhan case where the argument of the exponential is $\sim -t^{1/3}$.

This argument seems to suggest the independence of the strength of the disorder, which is irrelevant in regions in the neighborhood of $1/2$. Clearly the argument can be extended to higher dimensions and self-similar lattices.

VI. SPECTRAL ANALYSIS

A. Density of states

We shall here study the spectral properties of the TB equation associated with (2),
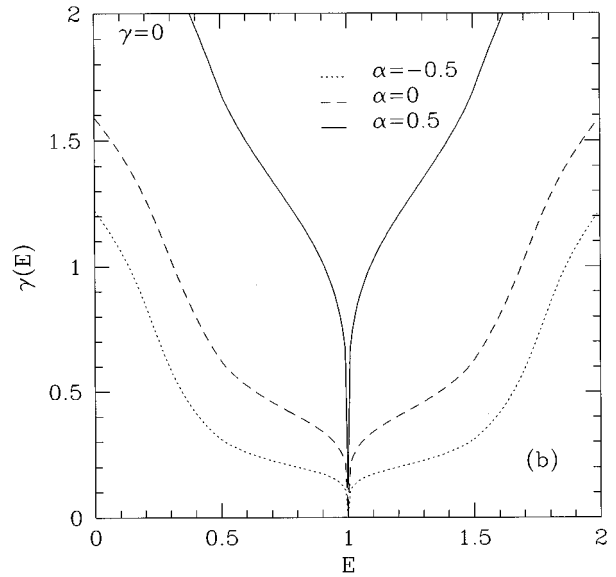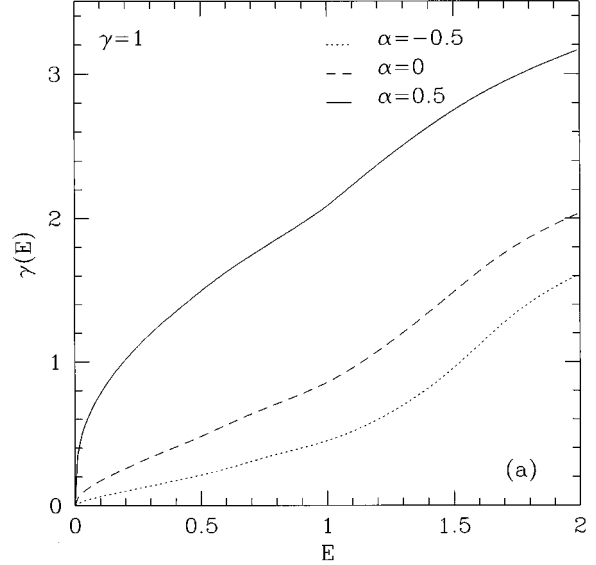

FIG. 5. Same as in Fig. 3, for the Lyapunov exponent $\gamma(E)$. A nonzero value of $\gamma(E)$ indicates that the eigenvalue E is localized.

$$\Omega_x(\gamma)\psi_x(E) - w_{x,x-1}\psi_{x-1}(E) - w_{x,x+1}\psi_{x+1}(E) = E\psi_x(E),$$
(32)

by studying the density of states and the localization properties. For $\gamma = 1$ the result is well known [17]. The Hamiltonian is positive definite as one can immediately check by defining the creation

$$(a^\dagger \psi)_x = \sqrt{w_{x,x-1}}\psi_{x-1} - \sqrt{w_{x,x+1}}\psi_x$$
(33)

and destruction

$$(a\psi)_x = \sqrt{w_{x+1,x}}\psi_{x+1} - \sqrt{w_{x,x+1}}\psi_x$$
(34)

operators and noting that $\mathscr{H}(1) = a^\dagger a$ [18]. Then for any $\gamma < 1$ we have that, using $\langle \psi | \varphi \rangle$ to indicate the usual scalar product

$$\langle \psi | \mathscr{H}(\gamma) | \psi \rangle > \langle \psi | \mathscr{H}(1) | \psi \rangle \geq 0,$$
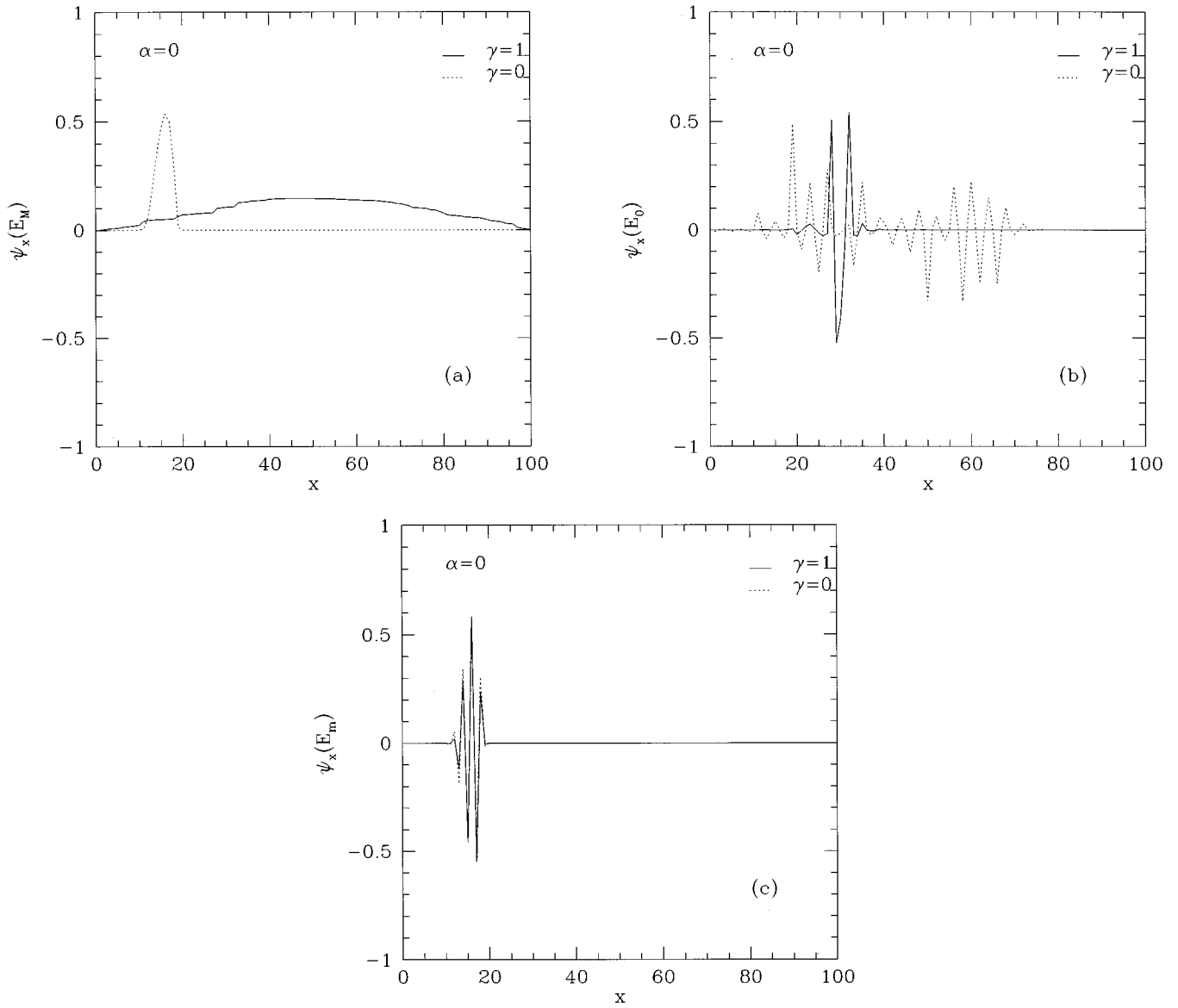(35)

FIG. 6. Spatial dependence of the eigenvectors $\psi_x(E)$ corresponding to the lower (E_M) (a), central (E_0), (b), and upper (E_m) (c) parts of the spectrum for the case $\alpha=0$ and $\gamma=0,1$.

and thus the Hamiltonian $\mathcal{H}(\gamma)$ is strictly positive for any $\gamma<1$.

A well known and efficient way to compute the spectrum of a one-dimensional disordered model is by means of the node-counting theorem [17]. By defining

$$U_x(E)=w_{x-1,x}\,\frac{\psi_x(E)}{\psi_{x-1}(E)} \qquad (36)$$

the following recursion can be easily found from (32):

$$U_{x+1}(E)=[\Omega_x(\gamma)-E]-\frac{w^2_{x,x-1}}{U_x(E)}. \qquad (37)$$

By calculating the number of times the quantity $U_x(E)$ changes sign, the integrated density of states

$$M(E)=\int_{-\infty}^{E} dE'\,N(E') \qquad (38)$$

[rather than the density of states $N(E)$] can be computed. The validity of the numerical procedure can be tested against the periodic case where the exact result

$$M(E)=\frac{1}{2}-\frac{1}{\pi}\arcsin(1-E) \qquad (39)$$

with $E\in[0,2]$ can be easily derived. The case $\gamma=1$ was analytically solved by Stephen and Kariotis [12]. They found in the $E\to0$ limit

$$M(E)=\begin{cases} E^{(1-\alpha)/(2-\alpha)} & \text{for } 0<\alpha<1 \\ E^{1/2}/\sqrt{|\ln\sqrt{E}|} & \text{for } \alpha=0. \end{cases} \qquad (40)$$

Our numerical results for $\alpha=0.5$ and 0 reproduce very well the theoretical prediction. Indeed, we find 0.34 ± 0.01 and 0.45 ± 0.01 for the two cases where the theoretical values are $1/3$ and $1/2$, respectively. The $\alpha=0$ case is less precise due to the logarithmic corrections. Our results are based on an av-
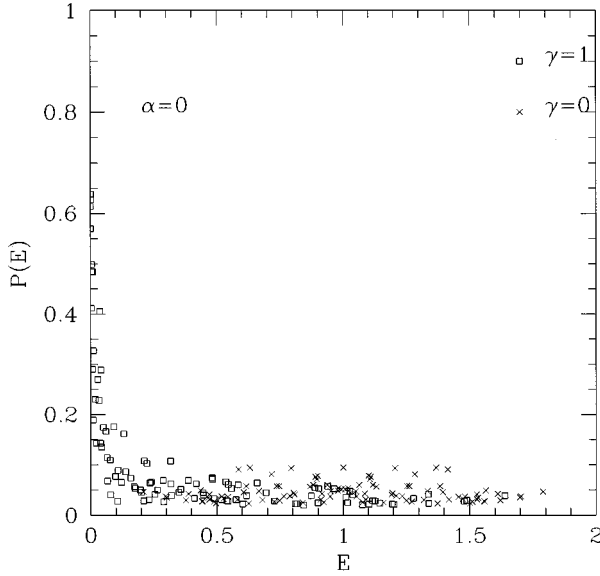
and

$$\Psi_x(E) = \begin{pmatrix} \psi_{x-1}(E) \\ \psi_x(E) \end{pmatrix}. \qquad (45)$$

Upon iteration Eq. (43) becomes, if N is the number of sites,

$$\Psi_N(E) = \left(\prod_{x=0}^{N-1} T_x(E) \right) \Psi_0(E), \qquad (46)$$

where we have taken the initial vector as

$$\Psi_0(E) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad (47)$$

The Lyapunov exponent is then defined as

$$\gamma(E) = \lim_{N \to \infty} \frac{1}{N} \ln \|\Psi_N(E)\|. \qquad (48)$$

Again we tested our algorithm in the case $\gamma = 1$ where an exact result [12] gives for $E \to 0$

$$\gamma(E) = \begin{cases} E^{(1-\alpha)/(2-\alpha)} & \text{for } 0 < \alpha < 1 \\ E^{1/2}/\sqrt{|\ln \sqrt{E}|} & \text{for } \alpha = 0. \end{cases} \qquad (49)$$

Our results are 0.34 ± 0.01 (to be compared with $1/3$) and 0.53 ± 0.02 (to be compared with $1/2$). For $E=0$ the Lyapunov exponent is zero, as it should be since it corresponds to the stationary state of the master equation (1) [19].

In the $\gamma = 0$ case, instead, we found that $\gamma(0) = c(\alpha)$, that is, the ground state is *localized* (see Fig. 5). The spectrum is symmetric around $E = 1$ and this stems from the fact that the matrix $\mathcal{H}(0) - 1$ is traceless. Again all the intermediate cases $(0 < \gamma < 1)$ follow this behavior with a shift of the center of the spectrum which depends on the value of α.

C. Exact diagonalization

As a cross check for the results of Secs. VI A and VI B, we direct-diagonalized the matrix in (3) for both $\gamma = 1$ and $\gamma = 0$ by using standard IMSL package routines for small matrices ($N = 100$). Although a direct quantitative comparison with the previous results is out of the question since much higher sizes would be necessary, these results provide a qualitative understanding of the main difference in the spectrum of (3) in the cases $\gamma = 1$ and 0. In Fig. 6 the spatial dependence of the eigenvector $\psi_x(E)$ is displayed for $\gamma = 0,1$ for an energy in the lower (a), central (b), and upper (c) parts of the spectrum. The chosen value for the strength of the disorder was $\alpha = 0$, but no qualitative change in the behavior is found for other values of the disorder in the relevant case $\gamma = 0$. Correctly the ground state $E = 0$ for $\gamma = 1$ is extended as we argued before.

The $\gamma = 0$ ground state is very localized consistently with our previous findings from the Lyapunov exponent.

Another general probe to test the localization properties is to compute the inverse participation ratio (IPR)

$$p(E) = \frac{1}{N} \frac{(\sum_x |\psi_x(E)|^2)^2}{\sum_x |\psi_x(E)|^4}, \qquad (50)$$

FIG. 7. Inverse participation ratio (IPR) defined in (50) as function of the eigenvalue E for $\alpha = 0$ and $\gamma = 0,1$. Values close to 1 indicate that the state is extended, while values close to 0 ($\sim 1/N$) mean that the corresponding eigenvalue E is localized.

erage of five different configurations with $N = 2^{18}$ sites. We found that the results changed very little from one configuration to another [this is tantamount to saying that $N(E)$ is self-averaging] and thus this average was sufficient for our purposes.

The behavior for $\gamma = 0$ is *qualitatively* different (see Fig. 3). We find for $E \to 0$ (see Fig. 4) a Lifshitz tail singularity:

$$M(E) = \exp(-1/E) \qquad (41)$$

for all three cases $\alpha = -0.5, 0, 0.5$. As expected on the basis of the result of Sec. III, all three values of the disorder give the same quantitative results. Fluctuations in the tail region grow as the disorder becomes stronger. We have been unable to give a theoretical derivation for this behavior. It should be noted that the EMA of Sec. IV yields

$$N(E) = \frac{1}{\pi} \frac{1}{\sqrt{2E - E^2 - c}} \qquad (42)$$

for $E \in [1 - 2\mu_a, 1 + 2\mu_a]$ and 0 otherwise with $c = 1 - 4\mu_a^2$. With $c = 0$ this is just the well known result for a periodic one-dimensional lattice. Again the EMA fails to give qualitatively correct predictions.

B. Lyapunov exponent

Equation (32) can be cast in a transfer matrix form,

$$\Psi_{x+1}(E) = T_x(E) \Psi_x(E), \qquad (43)$$

where we have defined the matrices

$$T_x(E) = \begin{pmatrix} 0 & 1 \\ -\dfrac{w_{x,x-1}}{w_{x,x+1}} & \dfrac{\Omega_x(\gamma) - E}{w_{x,x+1}} \end{pmatrix} \qquad (44)$$

where E and $\psi_x(E)$ are the eigenvalues and eigenvectors of Eq. (32) and N is the number of sites on the lattice. In this notation, then a state E is extended when $p(E) \sim 1$ while is localized whenever $p(E) \sim 1/N \ll 1$. The results are shown in Fig. 7 and are consistent with the previous picture.

VII. CONCLUSIONS

In this paper we have presented a detailed investigation of the properties of a one-dimensional disorder model for diffusion where at each site a fraction of the initial particles can disappear with random probability depending on the random adjacent barriers. Our work combines numerical and analytical techniques. The main results of this investigation can be summarized as follows.

(1) We showed that the trapped case ($\gamma < 1$) is qualitatively different from the conserved ($\gamma = 1$) counterpart and that a simple mean-field type of approach is not able to capture the large fluctuation introduced by the nonconservation of the probability.

(2) We found that the trapped case appears to be completely disorder dominated, thus suggesting a superuniversal behavior independent of the disorder (which is *not* the case for the conserved problem). We provided an intuitive explanation based on a heuristic derivation of the survival probability of why this may occur.

We believe that more analytical work would be necessary for a complete understanding of the particular features appearing in the model considered in this paper. Work in this direction is ongoing and will be reported in the future.

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